

PATHWISE UNIQUENESS FOR SINGULAR SDEs DRIVEN BY STABLE PROCESSES *

E. Priola

Abstract

We prove pathwise uniqueness for stochastic differential equations driven by non-degenerate symmetric α -stable Lévy processes with values in \mathbb{R}^d having a bounded and β -Hölder continuous drift term. We assume $\beta > 1 - \alpha/2$ and $\alpha \in [1, 2)$. The proof requires analytic regularity results for associated integro-differential operators of Kolmogorov type. We also study differentiability of solutions with respect to initial conditions and the homeomorphism property.

1 Introduction

In this paper we prove a pathwise uniqueness result for the following SDE

$$X_t = x + \int_0^t b(X_s) ds + L_t, \quad x \in \mathbb{R}^d, \quad t \geq 0, \quad (1.1)$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is bounded and β -Hölder continuous and $L = (L_t)$ is a non-degenerate d -dimensional symmetric α -stable Lévy process, $d \geq 1$.

Currently, there is a great interest in understanding pathwise uniqueness for SDEs when b is not Lipschitz continuous or, more generally, when b is singular enough so that the corresponding deterministic equation (1.1) with $L = 0$ is not well-posed. A remarkable result in this direction was proved by Veretennikov in [24] (see also [27] for $d = 1$). He was able to prove uniqueness when $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is only Borel and bounded and L is a standard d -dimensional Wiener process. This result has been generalized in various directions in [9], [13], [25], [6], [7], [5], [8].

The situation changes when L is not a Wiener process but is a symmetric α -stable process, $\alpha \in (0, 2)$. Indeed, when $d = 1$ and $\alpha < 1$, Tanaka, Tsuchiya and Watanabe prove in [23, Theorem 3.2] that even a bounded and β -Hölder continuous b is not enough to ensure pathwise uniqueness if $\alpha + \beta < 1$ (they consider drifts like $b(x) = \text{sign}(x)(|x|^\beta \wedge 1)$ and initial

*2010 Mathematics Subject Classification. Primary 60H10, 34F05. Secondary 60J75, 35B65. Supported by the M.I.U.R. research project Prin 2008 “Deterministic and stochastic methods in the study of evolution problems”.

condition $x = 0$). On the other hand, when $d = 1$ and $\alpha \geq 1$, they show pathwise uniqueness for any continuous and bounded b .

In this paper we prove pathwise uniqueness in any dimension $d \geq 1$, assuming that $\alpha \geq 1$ and b is bounded and β -Hölder continuous with $\beta > 1 - \alpha/2$. Our proof is different from the one in [23] and is inspired by [7]. The assumptions on the α -stable Lévy process L which we consider are collected in Section 2 (see in particular Hypothesis 1). Here we only mention two significant examples which satisfy our hypotheses. The first is when $L = (L_t)$ is a standard α -stable process (symmetric and rotationally invariant), i.e., the characteristic function of the random variable L_t is

$$E[e^{i\langle L_t, u \rangle}] = e^{-tc_\alpha |u|^\alpha}, \quad u \in \mathbb{R}^d, \quad t \geq 0, \quad (1.2)$$

where c_α is a positive constant. The second example is $L = (L_t^1, \dots, L_t^d)$, where L^1, \dots, L^d are independent one-dimensional symmetric stable processes of index α . In this case

$$E[e^{i\langle L_t, u \rangle}] = e^{-tk_\alpha(|u_1|^\alpha + \dots + |u_d|^\alpha)}, \quad u \in \mathbb{R}^d, \quad t \geq 0, \quad (1.3)$$

where k_α is a positive constant. Martingale problems for SDEs driven by (L_t^1, \dots, L_t^d) have been recently studied (see [3] and references therein).

We prove the following result.

Theorem 1.1. *Let L be a symmetric α -stable process with $\alpha \in [1, 2)$, satisfying Hypothesis 1 (see Section 2). Assume that $b \in C_b^\beta(\mathbb{R}^d; \mathbb{R}^d)$ for some $\beta \in (0, 1)$ such that*

$$\beta > 1 - \frac{\alpha}{2}.$$

Then pathwise uniqueness holds for equation (1.1). Moreover, let $X^x = (X_t^x)$ be the solution starting at $x \in \mathbb{R}^d$. We have:

(i) for any $t \geq 0$, $p \geq 1$, there exists a constant $C(t, p) > 0$ (depending also on α, β and $L = (L_t)$) such that

$$E\left[\sup_{0 \leq s \leq t} |X_s^x - X_s^y|^p\right] \leq C(t, p) |x - y|^p, \quad x, y \in \mathbb{R}^d; \quad (1.4)$$

(ii) for any $t \geq 0$, the mapping: $x \mapsto X_t^x$ is a homeomorphism from \mathbb{R}^d onto \mathbb{R}^d , P -a.s.;

(iii) for any $t \geq 0$, the mapping: $x \mapsto X_t^x$ is a C^1 -function on \mathbb{R}^d , P -a.s..

All these assertions require that L is non-degenerate. Estimate (1.4) replaces the standard Lipschitz-estimate which holds without expectation E when b is Lipschitz continuous. Assertion (ii) is the so-called homeomorphism property of solutions (we refer to [1], [19] and [14]; see also [26] for the case of Log-Lipschitz coefficients). Note that existence of strong solutions for (1.1) follows easily by a compactness argument (see the comment before Lemma 4.1). We mention that existence of weak solutions when b is only measurable and bounded is investigated in [15]. Since $C_b^{\beta'}(\mathbb{R}^d, \mathbb{R}^d) \subset$

$C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$ when $0 < \beta \leq \beta'$, our uniqueness result holds true for any $\alpha \geq 1$ when $\beta \in (1/2, 1)$. Theorem 1.1 implies the existence of a stochastic flow for (1.1) (see Remark 4.4) and gives a partial answer to a question posed by L. Mytnik.

The proof of the main result is given in Section 4. As in [7] our method is based on an Itô-Tanaka trick which requires suitable analytic regularity results. Such results are proved in Section 3. They provide global Schauder estimates for the following resolvent equation on \mathbb{R}^d

$$\lambda u - \mathcal{L}u - b \cdot Du = g, \quad (1.5)$$

where $\lambda > 0$ and $g \in C_b^\beta(\mathbb{R}^d)$ are given and we assume $\alpha \geq 1$ and $\alpha + \beta > 1$. Here \mathcal{L} is the generator of the Lévy process L (see (2.5), [1, Section 6.7] and [21, Section 31]). If L satisfies (1.2) then \mathcal{L} coincides with the fractional Laplacian $-(-\Delta)^{\alpha/2}$ on infinitely differentiable functions f with compact support (see [21, Example 32.7]), i.e., for any $x \in \mathbb{R}^d$,

$$-(-\Delta)^{\alpha/2} f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x) - 1_{\{|y| \leq 1\}} y \cdot Df(x)) \frac{\tilde{c}_\alpha}{|y|^{d+\alpha}} dy. \quad (1.6)$$

Schauder estimates are simpler to prove when $\alpha > 1$. In such a case, assuming in addition that $\mathcal{L} = -(-\Delta)^{\alpha/2}$, i.e., L is a standard α -stable process, these estimates can be deduced from the theory of fractional powers of sectorial operators (see [16]). We also mention [2, Section 7.3] where Schauder estimates are proved when $\alpha > 1$ and \mathcal{L} has the form (1.6) but with variable coefficients, i.e., $\tilde{c}_\alpha = \tilde{c}_\alpha(x, y)$. The limit case $\alpha = 1$ in (1.5) requires a special attention even for the fractional Laplacian $\mathcal{L} = -(-\Delta)^{1/2}$. Indeed in this case \mathcal{L} is of the “same order” of $b \cdot D$. To treat $\alpha = 1$, we use a localization procedure which is based on Theorem 3.3 where Schauder estimates are proved in the case of $b(x) = k$, for any $x \in \mathbb{R}^d$, showing that the Schauder constant is independent of k (see also Remark 3.5).

In order to prove Theorem 1.1, in Section 4 we apply Itô’s formula to $u(X_t)$, where $u \in C_b^{\alpha+\beta}$ comes from Schauder estimates for (1.5) when $g = b$ (in such case (1.5) must be understood componentwise). This is needed to perform the Itô-Tanaka trick and find a new equation for X_t in which the singular term $\int_0^t b(X_s) ds$ of (1.1) is replaced by more regular terms. Then uniqueness and (1.4) follow by L^p -estimates for stochastic integrals. Such estimates require the deterministic Lemma 4.1 and the condition $\alpha/2 + \beta > 1$. In addition, properties (ii) and (iii) are obtained transforming (1.1) into a form suitable for applying the results in [14].

We will use the letter c or C with subscripts for finite positive constants whose precise value is unimportant; the constants may change from proposition to proposition.

2 Preliminaries and notation

General references for this section are [1], [20, Chapter 2], [21] and [28].

Let $\langle u, v \rangle$ (or $u \cdot v$) be the euclidean inner product between u and $v \in \mathbb{R}^d$, for any $d \geq 1$; moreover $|u| = \langle u, u \rangle^{1/2}$. If $D \subset \mathbb{R}^d$ we denote by 1_D the indicator function of D . The Borel σ -algebra of \mathbb{R}^d will be indicated by $\mathcal{B}(\mathbb{R}^d)$. All the measures considered in the sequel will be positive and Borel. A measure γ on \mathbb{R}^d is called symmetric if $\gamma(D) = \gamma(-D)$, $D \in \mathcal{B}(\mathbb{R}^d)$.

Let us fix $\alpha \in (0, 2)$. In (1.1) we consider a d -dimensional *symmetric α -stable process* $L = (L_t)$, $d \geq 1$, defined on a fixed stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and \mathcal{F}_t -adapted; the stochastic basis satisfies the usual assumptions (see [1, page 72]). Recall that L is a Lévy process (i.e., it is continuous in probability, it has stationary increments, càdlàg trajectories, $L_t - L_s$ is independent of \mathcal{F}_s , $0 \leq s \leq t$, and $L_0 = 0$) with the additional property that its characteristic function is given by

$$E[e^{i\langle L_t, u \rangle}] = e^{-t\psi(u)}, \quad \psi(u) = - \int_{\mathbb{R}^d} \left(e^{i\langle u, y \rangle} - 1 - i\langle u, y \rangle 1_{\{|y| \leq 1\}}(y) \right) \nu(dy), \quad (2.1)$$

$u \in \mathbb{R}^d$, $t \geq 0$, where ν is a measure such that

$$\nu(D) = \int_{\mathbb{S}} \mu(d\xi) \int_0^\infty 1_D(r\xi) \frac{dr}{r^{1+\alpha}}, \quad D \in \mathcal{B}(\mathbb{R}^d), \quad (2.2)$$

for some symmetric, non-zero finite measure μ concentrated on the unitary sphere $\mathbb{S} = \{y \in \mathbb{R}^d : |y| = 1\}$ (see [21, Theorem 14.3]).

The measure ν is called the Lévy (intensity) measure of L and (2.1) is a special case of the Lévy-Khintchine formula. The measure ν is a σ -finite measure on \mathbb{R}^d such that $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (1 \wedge |y|^2) \nu(dy) < \infty$, with $1 \wedge |\cdot| = \min(1, |\cdot|)$. Note that formula (2.2) implies that (2.1) can be rewritten as

$$\begin{aligned} \psi(u) &= - \int_{\mathbb{R}^d} (\cos(\langle u, y \rangle) - 1) \nu(dy) \\ &= - \int_{\mathbb{S}} \mu(d\xi) \int_0^\infty \frac{\cos(\langle u, r\xi \rangle) - 1}{r^{1+\alpha}} dr = c_\alpha \int_{\mathbb{S}} |\langle u, \xi \rangle|^\alpha \mu(d\xi), \quad u \in \mathbb{R}^d \end{aligned} \quad (2.3)$$

(see also [21, Theorem 14.13]). The measure μ is called the spectral measure of the stable process L . In this paper we make the following non-degeneracy assumption (cf. [22] and [21, Definition 24.16]).

Hypothesis 1. The support of the spectral measure μ is not contained in a proper linear subspace of \mathbb{R}^d .

It is not difficult to show that Hypothesis 1 is equivalent to the following assertion: there exists a positive constant C_α such that, for any $u \in \mathbb{R}^d$,

$$\psi(u) \geq C_\alpha |u|^\alpha. \quad (2.4)$$

Condition (2.4) is also assumed in [11, Proposition 2.1]. To see that (2.4) implies Hypothesis 1, we argue by contradiction: if $\text{Supp}(\mu) \subset (M \cap \mathbb{S})$ where M is the hyperplane containing all vectors orthogonal to some $u_0 \neq 0$, then $\psi(u_0) = 0$. To show the converse, note that Hypothesis 1 implies that for

any $v \in \mathbb{R}^d$ with $|v| = 1$, we have $\psi(v) > 0$ (indeed, otherwise, we would have $\mu(\{\xi \in \mathbb{S} : |\langle v, \xi \rangle| > 0\}) = 0$ which contradicts the hypothesis). By using a compactness argument, we deduce that (2.4) holds for any $u \in \mathbb{R}^d$ with $|u| = 1$. Then, writing, for any $u \in \mathbb{R}^d$, $u \neq 0$, $\int_{\mathbb{S}} |\langle u, \xi \rangle|^\alpha \mu(d\xi) = |u|^\alpha \int_{\mathbb{S}} |\langle \frac{u}{|u|}, \xi \rangle|^\alpha \mu(d\xi)$, we obtain easily (2.4).

The infinitesimal generator \mathcal{L} of the process L is given by

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x) - 1_{\{|y| \leq 1\}} \langle y, Df(x) \rangle) \nu(dy), \quad f \in C_c^\infty(\mathbb{R}^d), \quad (2.5)$$

where $C_c^\infty(\mathbb{R}^d)$ is the space of all infinitely differentiable functions with compact support (see [21, Section 31]). Let us come back on the examples of α -stable processes considered in Introduction which satisfy Hypothesis 1. The first is when L is a standard α -stable process, i.e., $\psi(u) = c_\alpha |u|^\alpha$. In this case ν has density $\frac{C_\alpha}{|x|^{d+\alpha}}$ with respect to the Lebesgue measure in \mathbb{R}^d . Moreover the spectral measure μ is the normalized surface measure on \mathbb{S} (i.e., μ gives a uniform distribution on \mathbb{S} ; see [20, Section 2.5] and [21, Theorem 14.14]).

The second example is $L = (L_t^1, \dots, L_t^d)$, see (1.3). In this case $\psi(u) = k_\alpha(|u_1|^\alpha + \dots + |u_d|^\alpha)$ and the Lévy measure ν is more singular since it is concentrated on the union of the coordinates axes, i.e., ν has density

$$c_1 \left(1_{\{x_2=0, \dots, x_d=0\}} \frac{1}{|x_1|^{1+\alpha}} + \dots + 1_{\{x_1=0, \dots, x_{d-1}=0\}} \frac{1}{|x_d|^{1+\alpha}} \right)$$

with respect to the Lebesgue measure. The spectral measure μ is a linear combination of Dirac measures, i.e. $\mu = \sum_{k=1}^d (\delta_{e_k} + \delta_{-e_k})$, where (e_k) is the canonical basis in \mathbb{R}^d . The generator is

$$\mathcal{L}f(x) = \sum_{k=1}^d \int_{\mathbb{R}} [f(x+se_k) - f(x) - 1_{\{|s| \leq 1\}} s \partial_{x_k} f(x)] \frac{c_\alpha}{|s|^{1+\alpha}} ds, \quad f \in C_c^\infty(\mathbb{R}^d).$$

Let us fix some notation on function spaces.

We define $C_b(\mathbb{R}^d; \mathbb{R}^k)$, $k, d \geq 1$, as set of all functions $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$ which are bounded and continuous. It is a Banach space endowed with the supremum norm $\|f\|_0 = \sup_{x \in \mathbb{R}^d} |f(x)|$, $f \in C_b(\mathbb{R}^d; \mathbb{R}^k)$. Moreover, $C_b^\beta(\mathbb{R}^d; \mathbb{R}^k)$, $\beta \in (0, 1)$, is the subspace of all β -Hölder continuous functions f , i.e., f verifies

$$[f]_\beta := \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty. \quad (2.6)$$

$C_b^\beta(\mathbb{R}^d; \mathbb{R}^k)$ is a Banach space with the norm $\|\cdot\|_\beta = \|\cdot\|_0 + [\cdot]_\beta$. When $\mathbb{R}^k = \mathbb{R}$, we set $C_b^\beta(\mathbb{R}^d; \mathbb{R}^k) = C_b^\beta(\mathbb{R}^d)$. Let $C_b^0(\mathbb{R}^d; \mathbb{R}^k) = C_b(\mathbb{R}^d; \mathbb{R}^k)$ and $[\cdot]_0 = \|\cdot\|_0$. For any $n \geq 1$, $\alpha \in [0, 1)$, we say that $f \in C_b^{n+\alpha}(\mathbb{R}^d)$ if $f \in C_b^{n+\alpha}(\mathbb{R}^d) \cap C_b^\alpha(\mathbb{R}^d)$ and, for all $j = 1, \dots, n$, the (Fréchet) derivatives $D^j f \in C_b^\alpha(\mathbb{R}^d; (\mathbb{R}^d)^{\otimes(j+1)})$. The space $C_b^{n+\alpha}(\mathbb{R}^d)$ is a Banach space endowed with the norm $\|f\|_{n+\alpha} = \|f\|_0 + \sum_{k=1}^n \|D^k f\|_0 + [D^n f]_\alpha$, $f \in C_b^{n+\alpha}(\mathbb{R}^d)$.

Remark 2.1. Hypothesis 1 (or condition (2.4)) is equivalent to the following Picard's type condition (see [17]): there exists $\alpha \in (0, 2)$ and $C_\alpha > 0$, such that the following estimate holds, for any $\rho > 0$, $u \in \mathbb{R}^d$ with $|u| = 1$,

$$\int_{\{|\langle u, y \rangle| \leq \rho\}} |\langle u, y \rangle|^2 \nu(dy) \geq C_\alpha \rho^{2-\alpha}.$$

The equivalence follows from the computation

$$\begin{aligned} \int_{\{|\langle u, y \rangle| \leq \rho\}} |\langle u, y \rangle|^2 \nu(dy) &= \int_{\mathbb{S}} |\langle u, \xi \rangle|^2 \mu(d\xi) \int_0^\infty 1_{\{|\langle u, \xi \rangle| \leq \frac{\rho}{r}\}} r^{1-\alpha} dr \\ &= \rho^{2-\alpha} \int_{\mathbb{S}} |\langle u, \xi \rangle|^2 \mu(d\xi) \int_{|\langle u, \xi \rangle|}^\infty \frac{ds}{s^{3-\alpha}} = \frac{\rho^{2-\alpha}}{2-\alpha} \int_{\mathbb{S}} |\langle u, \xi \rangle|^\alpha \mu(d\xi). \end{aligned}$$

The Picard's condition is usually imposed on the Lévy measure ν of a non-necessarily stable Lévy process L in order to ensure that the law of L_t , for any $t > 0$, has a C^∞ -density with respect to the Lebesgue measure.

3 Some analytic regularity results

In this section we prove existence of regular solutions to (1.5). This result will be achieved through Schauder estimates and will be important in Section 4 to prove uniqueness for (1.1).

We will use the following three properties of the α -stable process L (in the sequel μ_t denotes the law of L_t , $t \geq 0$).

- (a) $\mu_t(A) = \mu_1(t^{-1/\alpha}A)$, for any $A \in \mathcal{B}(\mathbb{R}^d)$, $t > 0$ (this scaling property follows from (2.1) and (2.3));
- (b) μ_t has a density p_t with respect to the Lebesgue measure, $t > 0$; moreover $p_t \in C^1(\mathbb{R}^d)$ and its spatial derivative $Dp_t \in L^1(\mathbb{R}^d, \mathbb{R}^d)$ (this is a consequence of Hypothesis 1);
- (c) for any $\sigma > \alpha$, we have by (2.2)

$$\int_{\{|x| \leq 1\}} |x|^\sigma \nu(dx) < \infty. \quad (3.1)$$

The fact that (b) holds can be deduced by an argument of [22, Section 3]. Actually, Hypothesis 1 implies the following stronger result.

Lemma 3.1. *For any $\alpha \in (0, 2)$, $t > 0$, the density $p_t \in C^\infty(\mathbb{R}^d)$ and all derivatives $D^k p_t$ are integrable on \mathbb{R}^d , $k \geq 1$.*

Proof. We only show that $p_t \in C^\infty(\mathbb{R}^d)$ and $Dp_t \in L^1(\mathbb{R}^d, \mathbb{R}^d)$, following [22]; arguing in a similar way one can obtain the full assertion. By (2.4), we know that $e^{-t\psi(u)} \leq e^{ct|u|^\alpha}$, $u \in \mathbb{R}^d$, and so by the inversion formula of Fourier transform (see [21, Proposition 2.5]) μ_t has a density $p_t \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$,

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x, z \rangle} e^{-t\psi(z)} dz, \quad x \in \mathbb{R}^d, \quad t > 0. \quad (3.2)$$

Note that (a) implies that $p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha} x)$. Thanks to (2.4) one can differentiate infinitely many times under the integral sign and see that $p_t \in C^\infty(\mathbb{R}^d)$. Let us fix $j = 1, \dots, d$ and check that the partial derivative $\partial_{x_j} p_t \in L^1(\mathbb{R}^d)$. By the scaling property (a) it is enough to consider $t = 1$. By writing $\psi = \psi_1 + \psi_2$,

$$\psi_1(u) = - \int_{\{|y| \leq 1\}} (\cos(\langle u, y \rangle) - 1) \nu(dy), \quad \psi_2 = \psi - \psi_1,$$

$$\partial_{x_j} p_1(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle x, z \rangle} ((-iz_j) e^{-\psi_1(z)}) e^{-\psi_2(z)} dz, \quad x \in \mathbb{R}^d.$$

We find easily that $\psi_1 \in C^\infty(\mathbb{R}^d)$ and so, using also (2.4) we deduce that ψ_1 is in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. In particular, there exists $f_1 \in L^1(\mathbb{R}^d)$ such that the Fourier transform $\hat{f}_1(z) = (-iz_j) e^{-\psi_1(z)}$. On the other hand (see [21, Section 8]), there exists an infinitely divisible probability measure γ on \mathbb{R}^d such that the Fourier transform $\hat{\gamma}(z) = e^{-\psi_2(z)}$. By [21, Proposition 2.5] we infer that $\hat{f}_1 * \hat{\gamma} = \hat{f}_1 \cdot \hat{\gamma}$. By the inversion formula we deduce that $\partial_{x_j} p_1(x) = (f_1 * \gamma)(x)$ and this proves that $\partial_{x_j} p_1 \in L^1(\mathbb{R}^d)$. ■

Remark that (c) implies that the expression of $\mathcal{L}f$ in (2.5) is meaningful for any $f \in C_b^{1+\gamma}(\mathbb{R}^d)$ with $1 + \gamma > \alpha$. Indeed $\mathcal{L}f(x)$ can be decomposed into the sum of two integrals, over $\{|y| > 1\}$ and $\{|y| \leq 1\}$ respectively. The first integral is finite since f is bounded. To treat the second one, we can use the estimate

$$\begin{aligned} & |f(y+x) - f(x) - y \cdot Df(x)| \\ & \leq \int_0^1 |Df(x+ry) - Df(x)| |y| dr \leq \|Df\|_\gamma |y|^{1+\gamma}, \quad |y| \leq 1. \end{aligned} \tag{3.3}$$

Note that $\mathcal{L}f \in C_b(\mathbb{R}^d)$ if $f \in C_b^{1+\gamma}(\mathbb{R}^d)$ and $1 + \gamma > \alpha$.

The next result is a maximum principle. A related result is in [10, Section 4.5]. This will be used to prove uniqueness of solutions to (1.5) as well as to study existence.

Proposition 3.2. *Let $\alpha \in (0, 2)$. If $u \in C_b^{1+\gamma}(\mathbb{R}^d)$, $1 + \gamma > \alpha$, is a solution to $\lambda u - \mathcal{L}u - b \cdot Du = g$, with $\lambda > 0$ and $g \in C_b(\mathbb{R}^d)$, then*

$$\|u\|_0 \leq \frac{1}{\lambda} \|g\|_0, \quad \lambda > 0. \tag{3.4}$$

Proof. Since $-u$ solves the same equation of u with g replaced by $-g$, it is enough to prove that $u(x) \leq \frac{\|g\|_0}{\lambda}$, $x \in \mathbb{R}^d$. Moreover, possibly replacing u by $u - \inf_{x \in \mathbb{R}^d} u(x)$, we may assume that $u \geq 0$.

Now we show that there exists $c > 0$ such that, for any $\epsilon > 0$ we can find $u_\epsilon \in C_b^{1+\gamma}(\mathbb{R}^d)$ with $\|u_\epsilon\|_0 = \max_{x \in \mathbb{R}^d} |u_\epsilon(x)|$ and also

$$\|u - u_\epsilon\|_{1+\gamma} < \epsilon c.$$

To this purpose let $x_\epsilon \in \mathbb{R}^d$ be such that $u(x_\epsilon) > \|u\|_0 - \epsilon$ and take a test function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi(x_\epsilon) = 1$, $0 \leq \phi \leq 1$, and $\phi(x) = 0$ if $|x - x_\epsilon| \geq 1$. One checks that $u_\epsilon(x) = u(x) + 2\epsilon \phi(x)$ verifies the assumptions. Let us define the operator $\mathcal{L}_1 = \mathcal{L} + b \cdot D$ and write

$$\lambda u_\epsilon(x) - \mathcal{L}_1 u_\epsilon(x) = g(x) + \lambda(u_\epsilon(x) - u(x)) - \mathcal{L}_1(u_\epsilon - u)(x).$$

Let y_ϵ be one point in which u_ϵ attains its global maximum. Since clearly $\mathcal{L}_1 u_\epsilon(y_\epsilon) \leq 0$, we have (using also (3.3))

$$\lambda \|u_\epsilon\|_0 = \lambda u_\epsilon(y_\epsilon) \leq \|g\|_0 + C \|u - u_\epsilon\|_{1+\gamma} \leq \|g\|_0 + C_1 \epsilon.$$

Letting $\epsilon \rightarrow 0^+$, we get (3.4). ■

Next we prove Schauder estimates for (1.5) when b is constant. The case of $b \in C_b^\beta(\mathbb{R}^d, \mathbb{R}^d)$ will be treated in Theorem 3.4. We stress that the constant c in (3.6) is independent of $b = k$.

The condition $\alpha + \beta > 1$ which we impose is needed to have a regular C^1 -solution u . On the other hand, the next result holds more generally without the hypothesis $\alpha + \beta < 2$; this is imposed to simplify the proof and it is not restrictive in the study of pathwise uniqueness for (1.1).

Theorem 3.3. *Assume Hypothesis 1. Let $\alpha \in (0, 2)$ and $\beta \in (0, 1)$ be such that $1 < \alpha + \beta < 2$. Then, for any $\lambda > 0$, $k \in \mathbb{R}^d$, $g \in C_b^\beta(\mathbb{R}^d)$, there exists a unique solution $u = u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ to the equation*

$$\lambda u - \mathcal{L}u - k \cdot Du = g \tag{3.5}$$

on \mathbb{R}^d (\mathcal{L} is defined in (2.5)). In addition there exists a constant c independent of g , u , k and $\lambda > 0$ such that

$$\lambda \|u\|_0 + \lambda^{\frac{\alpha+\beta-1}{\alpha}} \|Du\|_0 + [Du]_{\alpha+\beta-1} \leq c \|g\|_\beta. \tag{3.6}$$

Proof. Equation (3.5) is meaningful for $u \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ with $\alpha + \beta > 1$ thanks to (3.3). Moreover, uniqueness follows from Proposition 3.2.

To prove the result, we use the semigroup approach as in [4]. To this purpose, we introduce the α -stable Markov semigroup (P_t) acting on $C_b(\mathbb{R}^d)$ and associated to $\mathcal{L} + k \cdot Du$, i.e.,

$$P_t f(x) = \int_{\mathbb{R}^d} f(z + tk) p_t(z - x) dz, \quad t > 0, \quad f \in C_b(\mathbb{R}^d), \quad x \in \mathbb{R}^d,$$

where p_t is defined in (3.2), and $P_0 = I$. Then we consider the bounded function $u = u_\lambda$,

$$u(x) = \int_0^\infty e^{-\lambda t} P_t g(x) dt, \quad x \in \mathbb{R}^d. \tag{3.7}$$

We are going to show that u belongs to $C_b^{\alpha+\beta}(\mathbb{R}^d)$, verifies (3.6) and solves (3.5).

I Part. We prove that $u \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ and that (3.6) holds.

First note that $\lambda\|u\|_0 \leq \|g\|_0$ since (P_t) is a contraction semigroup. Then, using the scaling property $p_t(x) = t^{-d/\alpha}p_1(t^{-1/\alpha}x)$, we arrive at

$$|DP_t f(x)| \leq \frac{t^{-1/\alpha}}{t^{d/\alpha}} \int_{\mathbb{R}^d} |f(z+tk)| |Dp_1(t^{-1/\alpha}z - t^{-1/\alpha}x)| dz \leq \frac{c_0\|f\|_0}{t^{1/\alpha}}, \quad (3.8)$$

$t > 0$, $f \in C_b(\mathbb{R}^d)$, where $c_0 = \|Dp_1\|_{L^1(\mathbb{R}^d)}$, and so we find the estimate

$$\|DP_t f\|_0 \leq \frac{c_0}{t^{1/\alpha}} \|f\|_0, \quad f \in C_b(\mathbb{R}^d), \quad t > 0. \quad (3.9)$$

By interpolation theory we know that $(C_b(\mathbb{R}^d), C_b^1(\mathbb{R}^d))_{\beta, \infty} = C_b^\beta(\mathbb{R}^d)$, $\beta \in (0, 1)$, see for instance [16, Chapter 1]; interpolating the previous estimate with the estimate $\|DP_t f\|_0 \leq \|Df\|_0$, $t \geq 0$, $f \in C_b^1(\mathbb{R}^d)$, we obtain

$$\|DP_t f\|_0 \leq \frac{c_1}{t^{(1-\beta)/\alpha}} \|f\|_\beta, \quad t > 0, \quad f \in C_b^\beta(\mathbb{R}^d), \quad (3.10)$$

with $c_1 = c_1(\alpha, \beta)$. In a similar way, we also find

$$\|D^2 P_t f\|_0 \leq \frac{c_2}{t^{(2-\beta)/\alpha}} \|f\|_\beta, \quad t > 0, \quad f \in C_b^\beta(\mathbb{R}^d). \quad (3.11)$$

Using (3.10) and the fact that $\frac{1-\beta}{\alpha} < 1$, we can differentiate under the integral sign in (3.7) and prove that there exists $Du(x) = Du_\lambda(x)$, $x \in \mathbb{R}^d$. Moreover Du_λ is bounded on \mathbb{R}^d and we have, for any $\lambda > 0$ with \tilde{c} independent of λ , u , k and g ,

$$\lambda^{\frac{\alpha+\beta-1}{\alpha}} \|Du\|_0 \leq \tilde{c} \|g\|_\beta$$

(we have used that $\int_0^\infty e^{-\lambda t} t^{-\sigma} dt = \frac{c}{\lambda^{1-\sigma}}$, for $\sigma < 1$ and $\lambda > 0$).

It remains to prove that $Du \in C_b^\theta(\mathbb{R}^d, \mathbb{R}^d)$, where $\theta = \alpha - 1 + \beta \in (0, 1)$. We proceed as in the proof of [2, Proposition 4.2] and [18, Theorem 4.2].

Using (3.10), (3.11) and the fact that $2 - \beta > \alpha$, we find, for any $x, x' \in \mathbb{R}^d$, $x \neq x'$,

$$\begin{aligned} |Du(x) - Du(x')| &\leq C \|g\|_\beta \left(\int_0^{|x-x'|^\alpha} \frac{1}{t^{(1-\beta)/\alpha}} dt + \int_{|x-x'|^\alpha}^\infty \frac{|x-x'|}{t^{(2-\beta)/\alpha}} dt \right) \\ &\leq c_3 \|g\|_\beta |x-x'|^\theta, \end{aligned}$$

and so $[Du]_{\alpha-1+\beta} \leq c_3 \|g\|_\beta$, where c_3 is independent of g , u , k and λ .

II Part. We prove that u solves (3.5), for any $\lambda > 0$.

We use the fact that the semigroup (P_t) is strongly continuous on the Banach space $C_0(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$ of all functions vanishing at infinity (endowed with $\|\cdot\|_0$; see [1, Section 6.7] and [21, Section 31]). Let $\mathcal{A} : D(\mathcal{A}) \subset C_0(\mathbb{R}^d) \rightarrow C_0(\mathbb{R}^d)$ be its generator. By [21, Theorem 31.5]) $C_0^2(\mathbb{R}^d) \subset D(\mathcal{A})$ and moreover $\mathcal{A}f = \mathcal{L}f + k \cdot Df$ if $f \in C_0^2(\mathbb{R}^d)$ (we say that f belongs to $C_0^2(\mathbb{R}^d)$ if $f \in C_b^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ and all its first and second partial derivatives belong to $C_0(\mathbb{R}^d)$).

We first show the assertion assuming in addition that $g \in C_0^2(\mathbb{R}^d)$.

It is easy to check that u belongs to $C_0^2(\mathbb{R}^d)$ as well. To this purpose, one can use the estimates $\|D^k P_t g\|_0 \leq \|D^k g\|_0$, $t \geq 0$, $k = 1, 2$, and the dominated convergence theorem. By the Hille-Yosida theorem we know that $u \in D(\mathcal{A})$ and $\lambda u - \mathcal{A}u = g$. Thus we have found that u solves (3.5).

Let us prove the assertion when $g \in C_b^2(\mathbb{R}^d)$.

Note that also $u \in C_b^2(\mathbb{R}^d)$. We consider a function $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\psi(0) = 1$ and introduce $g_n(x) = \psi(x/n)g(x)$, $x \in \mathbb{R}^d$, $n \geq 1$. It is clear that $g_n, u_n \in C_0^2(\mathbb{R}^d)$ (u_n is given in (3.7) when g is replaced by g_n). We know that

$$\lambda u_n(x) - \mathcal{L}u_n(x) - k \cdot Du_n(x) = g_n(x), \quad x \in \mathbb{R}^d. \quad (3.12)$$

It is easy to see that there exists $C > 0$ such that $\|g_n\|_2 \leq C$, $n \geq 1$, and moreover g_n and Dg_n converge pointwise to g and Dg respectively. It follows that also $\|u_n\|_2$ is uniformly bounded and moreover u_n and Du_n converge pointwise to u and Du respectively. Using also (3.3), we can apply the dominated convergence theorem and deduce that

$$\lim_{n \rightarrow \infty} \mathcal{L}u_n(x) = \mathcal{L}u(x), \quad x \in \mathbb{R}^d.$$

Passing to the limit in (3.12), we obtain that u is a solution to (3.5).

Let now $g \in C_b^\beta(\mathbb{R}^d)$.

Take any $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $0 \leq \phi \leq 1$ and $\int_{\mathbb{R}^d} \phi = 1$. Define $\phi_n(x) = n^d \phi(xn)$ and $g_n = g * \phi_n$. Note that $(g_n) \subset C_b^\infty(\mathbb{R}^d) = \cap_{k \geq 1} C_b^k(\mathbb{R}^d)$ and $\|g_n\|_\beta \leq \|g\|_\beta$, $n \geq 1$. Moreover, possibly passing to a subsequence still denoted by (g_n) , we may assume that

$$g_n \rightarrow g \quad \text{in } C^{\beta'}(K). \quad (3.13)$$

for any compact set $K \subset \mathbb{R}^d$ and $0 < \beta' < \beta$ (see page 37 in [12]). Let u_n be given in (3.7) when g is replaced by g_n . By the first part of the proof, we know that

$$\|u_n\|_{\alpha+\beta} \leq C \|g_n\|_\beta \leq C \|g\|_\beta,$$

where C is independent of n . It follows that, possibly passing to a subsequence still denoted with (u_n) , we have that $u_n \rightarrow u$ in $C^{\alpha+\beta'}(K)$, for any compact set $K \subset \mathbb{R}^d$ and $\beta' > 0$ such that $1 < \alpha + \beta' < \alpha + \beta$. Arguing as before, we can pass to the limit in $\lambda u_n(x) - \mathcal{L}u_n(x) - k \cdot Du_n(x) = g_n(x)$ and obtain that u solves (3.5). The proof is complete. \blacksquare

Now we extend Theorem 3.3 to the case in which b is Hölder continuous. We can only do this when $\alpha \geq 1$ (see also Remark 3.5). To prove the result when $\alpha = 1$ we adapt the localization procedure which is well known for second order uniformly elliptic operators with Hölder continuous coefficients (see [12]). This technique works in our situation since in estimate (3.6) the constant is independent of $k \in \mathbb{R}^d$.

The next proof requires the following interpolatory inequalities (see [12, page 40, (3.3.7)]); for any $t \in [0, 1)$, $0 \leq s \leq r < 1$, there exists $N = N(d, k, r, t)$ such that if $f \in C_b^{r+t}(\mathbb{R}^d, \mathbb{R}^k)$, then

$$[f]_{s+t} \leq N[f]_{r+t}^{s/r} [f]_t^{1-s/r}, \quad (3.14)$$

where $[f]_{s+t}$ is defined as in (2.6) if $0 < s+t < 1$, $[f]_0 = \|f\|_0$, $[f]_1 = \|Df\|_0$, and $[f]_{s+t} = [Df]_{s+t-1}$ if $1 < s+t < 2$. By (3.14) we deduce, for any $\epsilon > 0$,

$$[f]_{s+t} \leq \tilde{N}\epsilon^{r-s}[f]_{r+t} + \tilde{N}\epsilon^{-s}[f]_t, \quad f \in C_b^{r+t}(\mathbb{R}^d, \mathbb{R}^k). \quad (3.15)$$

Theorem 3.4. *Assume Hypothesis 1. Let $\alpha \geq 1$ and $\beta \in (0, 1)$ be such that $1 < \alpha + \beta < 2$. Then, for any $\lambda > 0$, $g \in C_b^\beta(\mathbb{R}^d)$, there exists a unique solution $u = u_\lambda \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ to the equation*

$$\lambda u - \mathcal{L}u - b \cdot Du = g \quad (3.16)$$

on \mathbb{R}^d . Moreover, for any $\omega > 0$, there exists $c = c(\omega)$, independent of g and u , such that

$$\lambda \|u\|_0 + [Du]_{\alpha+\beta-1} \leq c \|g\|_\beta, \quad \lambda \geq \omega. \quad (3.17)$$

Finally, we have $\lim_{\lambda \rightarrow \infty} \|Du_\lambda\|_0 = 0$.

Proof. Uniqueness and estimate $\lambda \|u\|_0 \leq \|g\|_0$, $\lambda > 0$, follow from the maximum principle (see Proposition 3.2). Moreover, the last assertion follows from (3.17) using (3.14). Indeed, with $t = 0$, $s = 1$, $r = \alpha + \beta$, we obtain, for $\lambda \geq \omega$,

$$[Du_\lambda]_0 = [u_\lambda]_1 \leq N[Du_\lambda]_{\alpha+\beta-1}^{\frac{1}{\alpha+\beta}} [u_\lambda]_0^{1-\frac{1}{\alpha+\beta}} \leq N\tilde{c} \lambda^{-\frac{\alpha+\beta-1}{\alpha+\beta}} \|g\|_\beta,$$

where $\tilde{c} = \tilde{c}(\omega)$. Letting $\lambda \rightarrow \infty$, we get the assertion.

Let us prove existence and estimate $[Du]_{\alpha+\beta-1} \leq c \|g\|_\beta$, for $\lambda \geq \omega$, with $\omega > 0$ fixed. We treat $\alpha > 1$ and $\alpha = 1$ separately.

I Part (the case $\alpha > 1$). In the sequel we will use the estimate

$$\|lf\|_\theta \leq \|l\|_0 \|f\|_\theta + \|f\|_0 [l]_\theta, \quad l, f \in C_b^\theta(\mathbb{R}^d), \quad \theta \in (0, 1). \quad (3.18)$$

Writing $\lambda u(x) - \mathcal{L}u(x) = g(x) + b(x) \cdot Du(x)$, and using (3.6) and (3.18), we obtain the following estimate (assuming that $u \in C_b^{\alpha+\beta}(\mathbb{R}^d)$ is a solution to (3.16))

$$\begin{aligned} [Du]_{\alpha+\beta-1} &\leq C \|g\|_\beta + C \|b \cdot Du\|_\beta \\ &\leq C \|g\|_\beta + C \|b\|_\beta \|Du\|_0 + C \|b\|_0 [Du]_\beta, \end{aligned} \quad (3.19)$$

where C is independent of $\lambda > 0$. Combining the interpolatory estimates (see (3.15) with $t = 0$, $s = 1 + \beta$, $r = \alpha + \beta$)

$$[Du]_\beta \leq \tilde{N}\epsilon^{\alpha-1} [Du]_{\alpha+\beta-1} + \tilde{N}\epsilon^{-(1+\beta)} \|u\|_0, \quad \epsilon > 0,$$

and $\|Du\|_0 \leq \tilde{N}\epsilon^{\alpha+\beta-1}[Du]_{\alpha+\beta-1} + \tilde{N}\epsilon^{-1}\|u\|_0$ (recall that $\alpha + \beta > 1 + \beta$) with the maximum principle, we get for ϵ small enough the a-priori estimate

$$\begin{aligned} \lambda\|u\|_0 + [Du]_{\alpha+\beta-1} &\leq c_1(\|g\|_\beta + C(\epsilon)\|u\|_0) \\ &\leq c_1(\|g\|_\beta + \frac{C(\epsilon)}{\omega}\|g\|_0) \leq C_1\|g\|_\beta, \end{aligned} \quad (3.20)$$

for any $\lambda \geq \omega$. Now to prove the existence of a $C_b^{\alpha+\beta}$ -solution, we use the continuity method (see, for instance, [12, Section 4.3]). Let us introduce

$$\lambda u(x) - \mathcal{L}u(x) - \delta b(x) \cdot Du(x) = g(x), \quad (3.21)$$

$x \in \mathbb{R}^d$, where $\delta \in [0, 1]$ is a parameter. Let us define $\Gamma = \{\delta \in [0, 1] : \text{there is a unique solution } u = u_\delta \in C_b^{\alpha+\beta}(\mathbb{R}^d), \text{ for any } g \in C_b^\beta(\mathbb{R}^d)\}$.

Clearly Γ is not empty since $0 \in \Gamma$. Fix $\delta_0 \in \Gamma$ and rewrite (3.21) as

$$\lambda u(x) - \mathcal{L}u(x) - \delta_0 b(x) \cdot Du(x) = g(x) + (\delta - \delta_0)b(x) \cdot Du(x).$$

Introduce the operator $S : C_b^{\alpha+\beta}(\mathbb{R}^d) \rightarrow C_b^{\alpha+\beta}(\mathbb{R}^d)$. For any $v \in C_b^{\alpha+\beta}(\mathbb{R}^d)$, $u = Sv$ is the unique $C_b^{\alpha+\beta}$ -solution to $\lambda u(x) - \mathcal{L}u(x) - \delta_0 b(x) \cdot Du(x) = g(x) + (\delta - \delta_0)b(x) \cdot Dv(x)$.

By using a-priori estimate (3.20), we find that $\|Sv_1 - Sv_2\|_{\alpha+\beta} \leq 2|\delta - \delta_0| \cdot \tilde{c}_1 \|b\|_\beta \|v_1 - v_2\|_{\alpha+\beta}$. By choosing $|\delta - \delta_0|$ small enough, S becomes a contraction and it has a unique fixed point which is the solution to (3.21). A compactness argument shows that $\Gamma = [0, 1]$. The assertion is proved.

II Part (the case $\alpha = 1$). As before, we establish the existence of a $C_b^{1+\beta}(\mathbb{R}^d)$ -solution, by using the continuity method. This requires an a-priori estimate (3.20) for $\alpha = 1$.

Let $u \in C_b^{1+\beta}(\mathbb{R}^d)$ be a solution. Let $r > 0$. Consider a function $\xi \in C_c^\infty(\mathbb{R}^d)$ such that $\xi(x) = 1$ if $|x| \leq r$ and $\xi(x) = 0$ if $|x| > 2r$.

Let now $x_0 \in \mathbb{R}^d$ and define $\rho(x) = \xi(x - x_0)$, $x \in \mathbb{R}^d$, and $v = u\rho$. One can easily check that

$$\mathcal{L}v(x) = \rho(x)\mathcal{L}u(x) + u(x)\mathcal{L}\rho(x) \quad (3.22)$$

$$+ \int_{\mathbb{R}^d} (\rho(x+y) - \rho(x))(u(x+y) - u(x)) \nu(dy), \quad x \in \mathbb{R}^d.$$

We have

$$\lambda v(x) - \mathcal{L}v(x) - b(x_0) \cdot Dv(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x), \quad x \in \mathbb{R}^d,$$

where

$$f_1(x) = \rho(x)g(x), \quad f_2(x) = (b(x) - b(x_0)) \cdot Dv(x),$$

$$f_3(x) = -u(x)[\mathcal{L}\rho(x) + b(x) \cdot D\rho(x)],$$

$$f_4(x) = - \int_{\mathbb{R}^d} (\rho(x+y) - \rho(x))(u(x+y) - u(x)) \nu(dy), \quad x \in \mathbb{R}^d.$$

By Theorem 3.3 we know that

$$[Dv]_\beta \leq C_1(\|f_1\|_\beta + \|f_2\|_\beta + \|f_3\|_\beta + \|f_4\|_\beta), \quad (3.23)$$

where the constant C_1 is independent of x_0 and λ . Let us consider the crucial term f_2 . By (3.18) we find

$$\|f_2\|_\beta \leq \left(\sup_{x \in B(x_0, 2r)} |b(x) - b(x_0)| \right) [Dv]_\beta + \|Dv\|_0 \|b\|_\beta.$$

Let us fix r small enough such that $C_1 \sup_{x \in B(x_0, 2r)} |b(x) - b(x_0)| < 1/2$. We get

$$[Dv]_\beta \leq 2C_1(\|f_1\|_\beta + \|Dv\|_0 \|b\|_\beta + \|f_3\|_\beta + \|f_4\|_\beta). \quad (3.24)$$

Note that $\|f_1\|_\beta \leq C(r) \|g\|_\beta$. Using again the interpolatory estimates (3.15) together with the maximum principle, we arrive at

$$[Dv]_\beta \leq C_2(\|g\|_\beta + \|f_3\|_\beta + \|f_4\|_\beta),$$

for any $\lambda \geq \omega$. Let us estimate f_4 . To this purpose we introduce the following non-local linear operator T

$$Tf(x) = \int_{\mathbb{R}^d} (\rho(x+y) - \rho(x))(f(x+y) - f(x)) \nu(dy), \quad f \in C_b^1(\mathbb{R}^d), \quad x \in \mathbb{R}^d.$$

One can easily check that T is continuous from $C_b^1(\mathbb{R}^d)$ into $C_b(\mathbb{R}^d)$ and from $C_b^{1+\beta}(\mathbb{R}^d)$ into $C_b^1(\mathbb{R}^d)$. To this purpose we only remark that, for any $x \in \mathbb{R}^d$,

$$\begin{aligned} |DTf(x)| &\leq 5 \|\rho\|_2 \|f\|_1 \left(\int_{\{|y| \leq 1\}} |y|^2 \nu(dy) + \int_{\{|y| > 1\}} \nu(dy) \right) \\ &+ 5 \|\rho\|_1 \|f\|_{1+\beta} \left(\int_{\{|y| \leq 1\}} |y|^{1+\beta} \nu(dy) + \int_{\{|y| > 1\}} \nu(dy) \right), \quad f \in C_b^{1+\beta}(\mathbb{R}^d). \end{aligned}$$

By interpolation theory we know that

$$\left(C_b^1(\mathbb{R}^d), C_b^{1+\beta}(\mathbb{R}^d) \right)_{\beta, \infty} = C_b^{1+\beta^2}(\mathbb{R}^d),$$

see [16, Chapter 1], and so we get that T is continuous from $C_b^{1+\beta^2}(\mathbb{R}^d)$ into $C_b^\beta(\mathbb{R}^d)$ (see [16, Theorem 1.1.6]). Since $f_4 = -Tu$, we obtain the estimate

$$\|f_4\|_\beta \leq C_3 \|u\|_{1+\beta^2}.$$

We have $\|f_4\|_\beta + \|f_3\|_\beta \leq c_3(r) \|u\|_{1+\beta^2}$ and so

$$[Dv]_\beta \leq C_4(\|g\|_\beta + \|u\|_{1+\beta^2}).$$

It follows that $[Du]_{C^\beta(B(x_0, r))} \leq C_4(\|g\|_\beta + \|u\|_{1+\beta^2})$, where $B(x_0, r)$ is the ball of center x_0 and radius $r > 0$. Since C_4 is independent of x_0 , we obtain

$$[Du]_\beta \leq C_4(\|g\|_\beta + \|u\|_{1+\beta^2}),$$

for any $\lambda \geq \omega$. Using again (3.15) and the maximum principle, we get the a-priori estimate (3.20) for $\alpha = 1$. Applying the continuity method we obtain the assertion. The proof is complete. \blacksquare

Remark 3.5. In contrast with Theorem 3.3, in Theorem 3.4 we can not show existence of $C_b^{\alpha+\beta}$ -solutions to (3.16) when $\alpha < 1$. The difficulty is evident from the a-priori estimate (3.19). Indeed, starting from

$$[Du]_{\alpha+\beta-1} \leq C\|g\|_\beta + C\|b\|_\beta\|Du\|_0 + C\|b\|_0[Du]_\beta,$$

we cannot continue, since $\alpha < 1$ gives $Du \in C_b^\theta$ with $\theta = \alpha + \beta - 1 < \beta$. Roughly speaking, when $\alpha < 1$, the perturbation term $b \cdot Du$ is of order larger than \mathcal{L} and so we cannot prove the desired a-priori estimates.

4 The main result

We briefly recall basic facts about Poisson random measures which we use in the sequel (see also [1], [14], [19], [28]). The Poisson random measure N associated with the process $L = (L_t)$ in (1.1) is defined by

$$N((0, t] \times U) = \sum_{0 < s \leq t} 1_U(\Delta L_s) = \#\{0 < s \leq t : \Delta L_s \in U\},$$

for any Borel set U in $\mathbb{R}^d \setminus \{0\}$, i.e., $U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $t > 0$. Here $\Delta L_s = L_s - L_{s-}$ denotes the jump size of L at time $s > 0$. The compensated Poisson random measure \tilde{N} is defined by $\tilde{N}((0, t] \times U) = N((0, t] \times U) - t\nu(U)$, where ν is given in (2.2). Recall that Lévy-Itô decomposition of the process L (see [1, Theorem 2.4.16] or [14, Theorem 2.7]). This says that

$$L_t = \hat{b}t + \int_0^t \int_{\{|x| \leq 1\}} x \tilde{N}(ds, dx) + \int_0^t \int_{\{|x| > 1\}} x N(ds, dx), \quad t \geq 0, \quad (4.1)$$

where $\hat{b} = E[L_1 - \int_0^1 \int_{\{|x| > 1\}} x N(ds, dx)]$. Note that in our case, since ν is symmetric, we have $\hat{b} = 0$.

The stochastic integral $\int_0^t \int_{\{|x| \leq 1\}} x \tilde{N}(ds, dx)$ (which is the compensated sum of small jumps) is an L^2 -martingale. The process $\int_0^t \int_{\{|x| > 1\}} x N(ds, dx) = \int_{(0, t]} \int_{\{|x| > 1\}} x N(ds, dx) = \sum_{0 < s \leq t, |\Delta L_s| > 1} \Delta L_s$ is a compound Poisson process.

Let $T > 0$. The predictable σ -field \mathcal{P} on $\Omega \times [0, T]$ is generated by all left-continuous adapted processes (defined on the same stochastic basis fixed in Section 2). Let $U \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$. In the sequel, we will always consider a $\mathcal{P} \times \mathcal{B}(U)$ -measurable mapping $F : [0, T] \times U \times \Omega \rightarrow \mathbb{R}^d$.

If $0 \notin \bar{U}$, then $\int_0^T \int_U F(s, x) N(ds, dx) = \sum_{0 < s \leq T} F(s, \Delta L_s) 1_U(\Delta L_s)$ as a random finite sum.

If $E \int_0^T ds \int_U |F(s, x)|^2 \nu(dx) < \infty$, then one can define the stochastic integral

$$Z_t = \int_0^t \int_U F(s, x) \tilde{N}(ds, dx), \quad t \in [0, T]$$

(here we do not assume $0 \notin \bar{U}$). The process $Z = (Z_t)$ is an L^2 -martingale with a càdlàg modification. Moreover, $E|Z_t|^2 = E \int_0^t ds \int_U |F(s, x)|^2 \nu(dx)$

(see [14, Lemma 2.4]). We will also use the following L^p -estimates (see [14, Theorem 2.11] or the proof of Proposition 6.6.2 in [1]); for any $p \geq 2$, there exists $c(p) > 0$ such that

$$\begin{aligned} E\left[\sup_{0 \leq s \leq t} |Z_s|^p\right] &\leq c(p)E\left[\left(\int_0^t ds \int_U |F(s, x)|^2 \nu(dx)\right)^{p/2}\right] \\ &+ c(p)E\left[\int_0^t ds \int_U |F(s, x)|^p \nu(dx)\right], \quad t \in [0, T] \end{aligned} \quad (4.2)$$

(the inequality is obvious if the right hand side is infinite).

Let us recall the concept of (*strong*) *solution* which we consider. A solution to the SDE (1.1) is a càdlàg \mathcal{F}_t -adapted process $X^x = (X_t^x)$ (defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ fixed in Section 2) which solves (1.1) P -a.s., for $t \geq 0$.

It is easy to show the existence of a solution to (1.1) using the fact that b is bounded and continuous. We may argue at ω fixed. Let us first consider $t \in [0, 1]$. By introducing $v(t) = X_t - L_t$, we get the equation

$$v(t) = x + \int_0^t b(v(s) + L_s) ds.$$

Approximating b with smooth drifts b_n we find solutions $v_n \in C([0, 1]; \mathbb{R}^d)$. By the Ascoli-Arzelà theorem, we obtain a solution to (1.1) on $[0, 1]$. The same argument works also on the time interval $[1, 2]$ with a random initial condition. Iterating this procedure we can construct a solution for all $t \geq 0$.

The proof of Theorem 1.1 requires some lemmas. We begin with a deterministic result.

Lemma 4.1. *Let $\gamma \in [0, 1]$ and $f \in C_b^{1+\gamma}(\mathbb{R}^d)$. Then for any $u, v \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, with $|x| \leq 1$, we have*

$$|f(u+x) - f(u) - f(v+x) + f(v)| \leq c_\gamma \|f\|_{1+\gamma} |u-v| |x|^\gamma, \quad \text{with } c_\gamma = 3^{1-\gamma} 2^\gamma.$$

Proof. For any $x \in \mathbb{R}^d$, $|x| \leq 1$, define the linear operator $T_x : C_b^1(\mathbb{R}^d) \rightarrow C_b^1(\mathbb{R}^d)$,

$$T_x f(u) = f(u+x) - f(u), \quad f \in C_b^1(\mathbb{R}^d), \quad u \in \mathbb{R}^d.$$

Since $\|T_x f\|_0 \leq \|Df\|_0 |x|$ and $\|D(T_x f)\|_0 \leq 2\|Df\|_0$, it follows that T_x is continuous and $\|T_x f\|_1 \leq (2 + |x|) \|f\|_1$, $f \in C_b^1(\mathbb{R}^d)$. Similarly, T_x is continuous from $C_b^2(\mathbb{R}^d)$ into $C_b^1(\mathbb{R}^d)$ and

$$\|T_x f\|_1 \leq 2|x| \|f\|_2, \quad f \in C_b^2(\mathbb{R}^d).$$

By interpolation theory $(C_b^1(\mathbb{R}^d), C_b^2(\mathbb{R}^d))_{\gamma, \infty} = C_b^{1+\gamma}(\mathbb{R}^d)$, see for instance [16, Chapter 1]; we deduce that, for any $\gamma \in [0, 1]$, T_x is continuous from $C_b^{1+\gamma}(\mathbb{R}^d)$ into $C_b^1(\mathbb{R}^d)$ (cf. [16, Theorem 1.1.6]) with operator norm less than or equal to $(2 + |x|)^{1-\gamma} (2|x|)^\gamma$.

Since $|x| \leq 1$, we obtain that $\|T_x f\|_1 \leq c_\gamma |x|^\gamma \|f\|_{1+\gamma}$, $f \in C_b^{1+\gamma}(\mathbb{R}^d)$. Now the assertion follows noting that, for any $u, v \in \mathbb{R}^d$,

$$|f(u+x) - f(u) - f(v+x) + f(v)| = |T_x f(u) - T_x f(v)| \leq \|DT_x f\|_0 |u - v|.$$

The proof is complete. \blacksquare

In the sequel we will consider the following resolvent equation on \mathbb{R}^d

$$\lambda u - \mathcal{L}u - Du \cdot b = b, \quad (4.3)$$

where $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is given in (1.1), the operator \mathcal{L} in (2.5) and $\lambda > 0$ (the equation must be understood componentwise, i.e., $\lambda u_i - \mathcal{L}u_i - b \cdot Du_i = b_i$, $i = 1, \dots, d$). The next two results hold for SDEs of type (1.1) when b is only continuous and bounded.

Lemma 4.2. *Let $\alpha \in (0, 2)$ and $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ in (1.1). Assume that, for some $\lambda > 0$, there exists a solution $u \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to (4.3) with $\gamma \in [0, 1]$, and moreover*

$$1 + \gamma > \alpha.$$

Let $X = (X_t)$ be a solution of (1.1) starting at $x \in \mathbb{R}^d$. We have, P -a.s., $t \geq 0$,

$$u(X_t) - u(x) \quad (4.4)$$

$$= x - X_t + L_t + \lambda \int_0^t u(X_s) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-})] \tilde{N}(ds, dx).$$

Proof. First note that the stochastic integral in (4.4) is meaningful thanks to the estimate

$$\begin{aligned} & E \int_0^t ds \int_{\mathbb{R}^d \setminus \{0\}} |u(X_{s-} + x) - u(X_{s-})|^2 \nu(dx) \\ & \leq 4t \|u\|_0 \int_{\{|x|>1\}} \nu(dx) + 2t \|u\|_1 \int_{\{|x|\leq 1\}} |x|^2 \nu(dx) < \infty. \end{aligned} \quad (4.5)$$

The assertion is obtained applying Itô's formula to $u(X_t)$ (for more details on Itô's formula see [1, Theorem 4.4.7] and [14, Section 2.3]).

A difficulty is that Itô's formula is usually stated for smooth functions $f \in C^2(\mathbb{R}^d)$. However, in the present situation in which L is a symmetric α -stable process, using (3.1), one can show that Itô's formula holds for any $f \in C_b^{1+\gamma}(\mathbb{R}^d)$. We give a proof of this fact.

Let $f \in C_b^{1+\gamma}(\mathbb{R}^d)$. We assume that $\gamma > 0$ (the proof with $\gamma = 0$ is similar). By mollifying f as in (3.13) we obtain a sequence $(f_n) \subset C_b^\infty(\mathbb{R}^d)$ such that $f_n \rightarrow f$ in $C^{1+\gamma'}(K)$, for any compact set $K \subset \mathbb{R}^d$ and $0 < \gamma' <$

γ . Moreover, $\|f_n\|_{1+\gamma} \leq \|f\|_{1+\gamma}$, $n \geq 1$. Let us fix $t > 0$. By Itô's formula we find, P -a.s.,

$$\begin{aligned} & f_n(X_t) - f_n(x) \\ &= \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [f_n(X_{s-} + x) - f_n(X_{s-})] \tilde{N}(ds, dx) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} [f_n(X_{s-} + x) - f_n(X_{s-}) - 1_{\{|x| \leq 1\}} x \cdot Df_n(X_{s-})] \nu(dx) \\ &+ \int_0^t b(X_s) \cdot Df_n(X_s) ds. \end{aligned} \quad (4.6)$$

It is not difficult to pass to the limit as $n \rightarrow \infty$; we show two arguments which are needed. To deal with the integral involving ν , one can apply the dominated convergence theorem, thanks to the following estimate similar to (3.3),

$$|f_n(X_{s-} + x) - f_n(X_{s-}) - x \cdot Df_n(X_{s-})| \leq c \|Df\|_\gamma |x|^{1+\gamma}, \quad |x| \leq 1$$

(recall that $\int_{\{|x| \leq 1\}} |x|^{1+\gamma} \nu(dx) < \infty$ since $1 + \gamma > \alpha$). In order to pass to the limit in the stochastic integral with respect to \tilde{N} , one uses the isometry formula

$$\begin{aligned} & E \left| \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [f_n(X_{s-} + x) - f(X_{s-} + x) - f_n(X_{s-}) + f(X_{s-})] \tilde{N}(ds, dx) \right|^2 \\ &= \int_0^t ds \int_{\{|x| \leq 1\}} E |f_n(X_{s-} + x) - f(X_{s-} + x) - f_n(X_{s-}) + f(X_{s-})|^2 \nu(dx) \\ &+ \int_0^t ds \int_{\{|x| > 1\}} E |f_n(X_{s-} + x) - f(X_{s-} + x) - f_n(X_{s-}) + f(X_{s-})|^2 \nu(dx). \end{aligned} \quad (4.7)$$

Arguing as in (4.5), since $\|f_n\|_{1+\gamma} \leq \|f\|_{1+\gamma}$, $n \geq 1$, we can apply the dominated convergence theorem in (4.7). Letting $n \rightarrow \infty$ in (4.7) we obtain 0. Finally, we can pass to the limit in probability in (4.6) and obtain Itô's formula when $f \in C_b^{1+\gamma}(\mathbb{R}^d)$.

Let now $u \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ as in the theorem. Noting that, for any $i = 1, \dots, d$,

$$\mathcal{L}u_i(y) = \int_{\mathbb{R}^d} [u_i(y + x) - u_i(y) - 1_{\{|x| \leq 1\}} x \cdot Du_i(y)] \nu(dx), \quad y \in \mathbb{R}^d,$$

and using that u solves (4.3), i.e., $\mathcal{L}u + b \cdot Du = \lambda u - b$, we can replace in the Itô formula for $u(X_t)$ the term

$$\begin{aligned} & \int_0^t \mathcal{L}u(X_s) ds + \int_0^t Du(X_s) b(X_s) ds \\ &= \sum_{i=1}^d \left(\int_0^t \mathcal{L}u_i(X_s) ds + \int_0^t Du_i(X_s) \cdot b(X_s) ds \right) e_i \end{aligned}$$

with $-\int_0^t b(X_s)ds + \lambda \int_0^t u(X_s)ds = x - X_t + L_t + \lambda \int_0^t u(X_s)ds$ and obtain the assertion. \blacksquare

The proof of Theorem 1.1 will be a consequence of the following result.

Theorem 4.3. *Let $\alpha \in (0, 2)$ and $b \in C_b(\mathbb{R}^d, \mathbb{R}^d)$ in (1.1). Assume that, for some $\lambda > 0$, there exists a solution $u = u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to the resolvent equation (4.3) with $\gamma \in [0, 1]$, such that $c_\lambda = \|Du_\lambda\|_0 < 1/3$. Moreover, assume that*

$$2\gamma > \alpha.$$

Then the SDE (1.1), for every $x \in \mathbb{R}^d$, has a unique solution (X_t^x) .

Moreover, assertions (i), (ii) and (iii) of Theorem 1.1 hold.

Proof. Note that $2\gamma > \alpha$ implies the condition $1 + \gamma > \alpha$ of Lemma 4.2.

We provide a direct proof of *pathwise uniqueness and assertion (i)*. This uses Lemmas 4.2 and 4.1 together with L^p -estimates for stochastic integrals (see (4.2)). Statements (ii) and (iii) will be obtained by transforming (1.1) in a form suitable for applying the results in [14, Chapter 3].

Let us fix $t > 0$, $p \geq 2$ and consider two solutions X and Y of (1.1) starting at x and $y \in \mathbb{R}^d$ respectively. Note that X_t is not in L^p if $p \geq \alpha$ (compare with [14, Theorem 3.2]) but the difference $X_t - Y_t$ is a bounded process. Pathwise uniqueness and (1.4) (for any $p \geq 1$) follow if we prove

$$E[\sup_{0 \leq s \leq t} |X_s - Y_s|^p] \leq C(t) |x - y|^p, \quad x, y \in \mathbb{R}^d, \quad (4.8)$$

with a positive constant $C(t)$ independent of x and y . Indeed in the special case of $x = y$ estimate (4.8) gives uniqueness of solutions.

We have from Lemma 4.2, P -a.s.,

$$\begin{aligned} X_t - Y_t &= [x - y] + [u(x) - u(y)] + [u(Y_t) - u(X_t)] \\ &+ \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} [u(X_{s-} + x) - u(X_{s-}) - u(Y_{s-} + x) + u(Y_{s-})] \tilde{N}(ds, dx) \\ &+ \lambda \int_0^t [u(X_s) - u(Y_s)] ds. \end{aligned} \quad (4.9)$$

Since $\|Du\|_0 \leq 1/3$, we have $|u(X_t) - u(Y_t)| \leq \frac{1}{3}|X_t - Y_t|$. It follows the estimate $|X_t - Y_t| \leq \frac{3}{2}\Lambda_1(t) + \frac{3}{2}\Lambda_2(t) + \frac{3}{2}\Lambda_3(t) + \frac{3}{2}\Lambda_4$, where

$$\Lambda_1(t) = \left| \int_0^t \int_{\{|x|>1\}} [u(X_{s-} + x) - u(X_{s-}) - u(Y_{s-} + x) + u(Y_{s-})] \tilde{N}(ds, dx) \right|,$$

$$\Lambda_2(t) = \lambda \int_0^t |u(X_s) - u(Y_s)| ds,$$

$$\Lambda_3(t) = \left| \int_0^t \int_{\{|x|\leq 1\}} [u(X_{s-} + x) - u(X_{s-}) - u(Y_{s-} + x) + u(Y_{s-})] \tilde{N}(ds, dx) \right|,$$

$\Lambda_4 = |x - y| + |u(x) - u(y)| \leq \frac{4}{3}|x - y|$. Note that, P -a.s.,

$$\sup_{0 \leq s \leq t} |X_s - Y_s|^p \leq C_p |x - y|^p + C_p \sum_{k=1}^3 \sup_{0 \leq s \leq t} \Lambda_k(s)^p.$$

The main difficulty is to estimate $\Lambda_3(t)$. Let us first consider the other terms. By the Hölder inequality

$$\sup_{0 \leq s \leq t} \Lambda_2(s)^p \leq c_1(p) t^{p-1} \int_0^t \sup_{0 \leq s \leq r} |X_s - Y_s|^p dr.$$

By (4.2) with $U = \{x \in \mathbb{R}^d : |x| > 1\}$ we find

$$\begin{aligned} & E\left[\sup_{0 \leq s \leq t} \Lambda_1(s)^p\right] \\ & \leq c(p) E\left[\left(\int_0^t ds \int_{\{|x|>1\}} |u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})|^2 \nu(dx)\right)^{p/2}\right] \\ & + c(p) E \int_0^t ds \int_{\{|x|>1\}} |u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})|^p \nu(dx). \end{aligned}$$

Using $|u(X_{s-}+x) - u(Y_{s-}+x) + u(Y_{s-}) - u(X_{s-})| \leq \frac{2}{3}|X_{s-} - Y_{s-}|$ and the Hölder inequality, we get

$$\begin{aligned} & E\left[\sup_{0 \leq s \leq t} \Lambda_1(s)^p\right] \leq C_1(p) (1 + t^{p/2-1}) \cdot \\ & \cdot \left(\int_{\{|x|>1\}} \nu(dx) + \left(\int_{\{|x|>1\}} \nu(dx) \right)^{p/2} \right) \int_0^t E\left[\sup_{0 \leq s \leq r} |X_s - Y_s|^p\right] dr. \end{aligned}$$

Let us treat $\Lambda_3(t)$. This requires the condition $2\gamma > \alpha$. By using (4.2) with $U = \{x \in \mathbb{R}^d : |x| \leq 1, x \neq 0\}$ and also Lemma 4.1, we get

$$\begin{aligned} & E\left[\sup_{0 \leq s \leq t} \Lambda_3(s)^p\right] \leq c(p) \|u\|_{1+\gamma}^p E\left[\left(\int_0^t ds \int_{\{|x|\leq 1\}} |X_s - Y_s|^2 |x|^{2\gamma} \nu(dx)\right)^{p/2}\right] \\ & + c(p) \|u\|_{1+\gamma}^p E \int_0^t ds \int_{\{|x|\leq 1\}} |X_s - Y_s|^p |x|^{\gamma p} \nu(dx). \end{aligned}$$

We obtain

$$\begin{aligned} & E\left[\sup_{0 \leq s \leq t} \Lambda_3(s)^p\right] \leq C_2(p) (1 + t^{p/2-1}) \|u\|_{1+\gamma}^p \cdot \\ & \cdot \left(\left(\int_{\{|x|\leq 1\}} |x|^{2\gamma} \nu(dx) \right)^{p/2} + \int_{\{|x|\leq 1\}} |x|^{\gamma p} \nu(dx) \right) \int_0^t E\left[\sup_{0 \leq s \leq r} |X_s - Y_s|^p\right] dr, \end{aligned}$$

where $\int_{\{|x|\leq 1\}} |x|^{\gamma p} \nu(dx) < +\infty$, since $p \geq 2$ and $2\gamma > \alpha$. Collecting the previous estimates, we arrive at

$$E\left[\sup_{0 \leq s \leq t} |X_s - Y_s|^p\right] \leq C_p |x - y|^p + C_4(p) (1 + t^{p-1}) \int_0^t E\left[\sup_{0 \leq s \leq r} |X_s - Y_s|^p\right] dr.$$

Applying the Gronwall lemma we obtain (4.8) with $C(t) = C_p \exp(C_4(p)(1+t^{p-1}))$. The assertion is proved.

Now we establish the *homeomorphism property (ii)* (cf. [14, Chapter 3], [1, Chapter 6] and [19, Section V.10]).

First note that, since $\|Du\|_0 < 1/3$, the classical Hadamard theorem (see [19, page 330]) implies that the mapping $\psi : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\psi(x) = x + u(x)$, $x \in \mathbb{R}^d$, is a C^1 -diffeomorphism from \mathbb{R}^d onto \mathbb{R}^d . Moreover, $D\psi^{-1}$ is bounded on \mathbb{R}^d and $\|D\psi^{-1}\|_0 \leq \frac{1}{1-c_\lambda} < \frac{3}{2}$ thanks to

$$D\psi^{-1}(y) = [I + Du(\psi^{-1}(y))]^{-1} = \sum_{k \geq 0} (-Du(\psi^{-1}(y)))^k, \quad y \in \mathbb{R}^d. \quad (4.10)$$

Let $r \in (0, 1)$ and introduce the SDE

$$Y_t = y + \int_0^t \tilde{b}(Y_s) ds \quad (4.11)$$

$$\int_0^t \int_{\{|z| \leq r\}} g(Y_{s-}, z) \tilde{N}(ds, dz) + \int_0^t \int_{\{|z| > r\}} g(Y_{s-}, z) N(ds, dz), \quad t \geq 0,$$

where $\tilde{b}(y) = \lambda u(\psi^{-1}(y)) - \int_{\{|z| > r\}} [u(\psi^{-1}(y) + z) - u(\psi^{-1}(y))] \nu(dz)$ and

$$g(y, z) = u(\psi^{-1}(y) + z) + z - u(\psi^{-1}(y)), \quad y \in \mathbb{R}^d, \quad z \in \mathbb{R}^d \setminus \{0\}.$$

Note that (4.11) is a SDE of the type considered in [14, Section 3.5]. Due to the Lipschitz condition, there exists a unique solution $Y^y = (Y_t^y)$ to (4.11). Moreover, using (4.4) and (4.1) with $\hat{b} = 0$, it is not difficult to show that

$$\psi(X_t^x) = Y_t^{\psi(x)}, \quad x \in \mathbb{R}^d, \quad t \geq 0. \quad (4.12)$$

Thanks to (4.12) to prove our assertion, it is enough to show the homeomorphism property for Y_t^y . To this purpose, we will apply [14, Theorem 3.10] to equation (4.11). Let us check its assumptions.

Clearly, \tilde{b} is Lipschitz continuous and bounded. Let us consider [14, condition (3.22)]. For any $y \in \mathbb{R}^d$, $z \in \mathbb{R}^d \setminus \{0\}$, $|g(y, z)| \leq |z|(1 + \|u\|_1) \leq K(z)$, where $K(z) = \frac{4}{3}|z|$ (recall that $\int_{|z| \leq 1} |z|^2 \nu(dz) < \infty$); further by Lemma 4.1 and (4.10) we have

$$|g(y, z) - g(y', z)| \leq L(z)|y - y'|, \quad y, y' \in \mathbb{R}^d, \quad \text{where } L(z) = C_1 \|u\|_{1+\gamma} |z|^\gamma,$$

$|z| \leq 1$, with $\int_{|z| \leq 1} L(z)^2 \nu(dz) < \infty$, since $2\gamma > \alpha$. Note that we may fix $r > 0$ small enough in (4.11) in order that $K(r) + L(r) < 1$ (according to [14, Section 3.5] this condition allows to deduce that equation (4.11) without $\int_0^t \int_{\{|z| > r\}} g(Y_{s-}, z) N(ds, dz)$ satisfies the homeomorphism property).

In order to get the homeomorphism property, it remains to check that, for any $z \in \mathbb{R}^d \setminus \{0\}$, the mapping:

$$y \mapsto y + g(y, z) \quad \text{is a homeomorphism from } \mathbb{R}^d \text{ onto } \mathbb{R}^d. \quad (4.13)$$

Let us fix z . To check the assertion, we will again apply the Hadamard theorem. We have

$$D_y g(y, z) = [Du(\psi^{-1}(y) + z) - Du(\psi^{-1}(y))] [D\psi^{-1}(y)]$$

and so by (4.10) (since $\|Du\|_0 < 1/3$) we get $\|D_y g(\cdot, z)\|_0 \leq \frac{2c_\lambda}{1-c_\lambda} < 1$. We have obtained (4.13). By [14, Theorem 3.10] the homeomorphism property for Y_t^y follows and this gives the assertion.

Now we show that, for any $t \geq 0$, the mapping: $x \mapsto X_t^x$ is of class C^1 on \mathbb{R}^d , P -a.s. (see (iii)).

We fix $t > 0$ and a unitary vector e_k of the canonical basis in \mathbb{R}^d . We will show that there exists, P -a.s., the partial derivative $\lim_{\lambda \rightarrow 0} \frac{X_t^{x+\lambda e_k} - X_t^x}{\lambda} = D_{e_k} X_t^x$ and, moreover, that the mapping $x \mapsto D_{e_k} X_t^x$ is continuous on \mathbb{R}^d , P -a.s..

Let us consider the process $Y^y = (Y_t^y)$ which solves the SDE (4.11). If we prove that the mapping $y \mapsto Y_t^y$ is of class C^1 on \mathbb{R}^d , P -a.s., then we have proved the assertion. Indeed, P -a.s.

$$D_{e_k} X_t^x = [D\psi^{-1}(Y_t^{\psi(x)})][DY_t^{\psi(x)}] D_{e_k} \psi(x), \quad x \in \mathbb{R}^d.$$

In order to apply [14, Theorem 3.4] we introduce the process (Z_t^y) which solves

$$Z_t = y + \lambda \int_0^t u(\psi^{-1}(Z_s)) ds + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} h(Z_{s-}, z) \tilde{N}(ds, dz), \quad (4.14)$$

$t \geq 0$, $y \in \mathbb{R}^d$, where

$$h(y, z) = u(\psi^{-1}(y) + z) - u(\psi^{-1}(y)) = g(y, z) - z$$

(adding L_t to (4.14), one gets (4.11)). Proving that $y \mapsto Z_t^y$ is of class C^1 on \mathbb{R}^d , P -a.s., is equivalent to show that $y \mapsto Y_t^y$ is of class C^1 on \mathbb{R}^d . Indeed, we have $\lim_{\lambda \rightarrow 0} \frac{Y_t^{y+\lambda e_k} - Y_t^y}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{Z_t^{y+\lambda e_k} - Z_t^y}{\lambda}$.

To prove the assertion for Z_t^y , it is enough to check that the SDE (4.14) verifies the assumptions of [14, Theorem 3.4]. These are, respectively, [14, conditions (3.1), (3.2), (3.8) and (3.9)]. Conditions (3.1) and (3.2) are easy to check. Indeed $\lambda u(\psi^{-1}(\cdot))$ is Lipschitz continuous on \mathbb{R}^d and, moreover, thanks to Lemma 4.1 and to the boundeness of $D\psi^{-1}$,

$$|h(y, z) - h(y', z)| \leq C(1_{\{|z| \leq 1\}} |z|^\gamma + 1_{\{|z| > 1\}}) |y - y'|, \quad z \in \mathbb{R}^d \setminus \{0\},$$

$y, y' \in \mathbb{R}^d$, with $\int_{\mathbb{R}^d} (1_{\{|z| \leq 1\}} |z|^\gamma + 1_{\{|z| > 1\}})^p \nu(dz) < \infty$, for any $p \geq 2$. In addition, $|h(y, z)| \leq L(z)$, $z \in \mathbb{R}^d \setminus \{0\}$, $y \in \mathbb{R}^d$, where, since $\|Du\|_0 < 1/3$,

$$L(z) = \frac{1}{3} 1_{\{|z| \leq 1\}} |z| + 2\|u\|_0 1_{\{|z| > 1\}} \quad \text{with} \quad \int_{\mathbb{R}^d} L(z)^p \nu(dz) < \infty, \quad p \geq 2.$$

Assumptions [14, (3.8) and (3.9)] are more difficult to check. They require that there exists some $\delta > 0$ such that (setting $l(x) = \lambda u(\psi^{-1}(x))$)

$$(1) \sup_{y \in \mathbb{R}^d} |Dl(y)| < \infty; \quad |Dl(y) - Dl(y')| \leq C|y - y'|^\delta, \quad y, y' \in \mathbb{R}^d.$$

$$(2) |D_y h(y, z)| \leq K_1(z); \quad |D_y h(y, z) - D_y h(y', z)| \leq K_2(z) |y - y'|^\delta, \quad (4.15)$$

for any $y, y' \in \mathbb{R}^d$, $z \in \mathbb{R}^d \setminus \{0\}$, with $\int_{\mathbb{R}^d} K_i(z)^p \nu(dz) < \infty$, for any $p \geq 2$, $i = 1, 2$. Such estimates are used in [14] in combination with the Kolmogorov continuity theorem to show the differentiability property.

Let us check (1) with $\delta = \gamma$, i.e., $Dl \in C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d)$. Since, for any $y \in \mathbb{R}^d$, $Dl(y) = \lambda Du(\psi^{-1}(y)) D\psi^{-1}(y)$, we find that Dl is bounded on \mathbb{R}^d . Moreover, thanks to the following estimate (cf. (3.18))

$$[Dl]_\gamma \leq \lambda \|Du\|_0 [D\psi^{-1}]_\gamma + \lambda [Du]_\gamma \|D\psi^{-1}\|_0^{1+\gamma},$$

in order to prove the assertion it is enough to show that $[D\psi^{-1}]_\gamma < \infty$. Recall that for $d \times d$ real matrices A and B , we have $(I + A)^{-1} - (I + B)^{-1} = (I + A)^{-1}(B - A)(I + B)^{-1}$ (if $(I + A)$ and $(I + B)$ are invertible). We obtain, using also that $D\psi^{-1}$ is bounded,

$$\begin{aligned} |D\psi^{-1}(y) - D\psi^{-1}(y')| &= |[I + Du(\psi^{-1}(y))]^{-1} - [I + Du(\psi^{-1}(y'))]^{-1}| \\ &\leq c_1 [Du]_\gamma |y - y'|^\gamma, \quad y, y' \in \mathbb{R}^d \end{aligned}$$

and the proof of (1) is complete with $\gamma = \delta$. Let us consider (2). Clearly, $D_y h(y, z) = [Du(\psi^{-1}(y) + z) - Du(\psi^{-1}(y))] D\psi^{-1}(y)$ verifies the first part of (2) with

$$K_1(z) = c_2 \|Du\|_\gamma (1_{\{|z| \leq 1\}} |z|^\gamma + 1_{\{|z| > 1\}}).$$

Let us deal with the second part of (2). We choose $\gamma' \in (0, \gamma)$ such that $2\gamma' > \alpha$ and first show that, for any $f \in C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d)$, we have

$$[T_x f]_{\gamma-\gamma'} \leq C[f]_\gamma |x|^{\gamma'}, \quad x \in \mathbb{R}^d, \quad (4.16)$$

where (as in Lemma 4.1) for any $x \in \mathbb{R}^d$, we define the mapping $T_x f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ as $T_x f(u) = f(x + u) - f(u)$, $u \in \mathbb{R}^d$. Using also (3.14) we get

$$[T_x f]_{\gamma-\gamma'} \leq N [T_x f]_\gamma^{\frac{\gamma-\gamma'}{\gamma}} [T_x f]_0^{1-\frac{\gamma-\gamma'}{\gamma}} \leq cN [f]_\gamma |x|^{\gamma(1-\frac{\gamma-\gamma'}{\gamma})} \leq cN |x|^{\gamma'} [f]_\gamma,$$

for any $x \in \mathbb{R}^d$. By (4.16) we will prove (2) with $\delta = \gamma - \gamma' > 0$.

First consider the case when $|z| \leq 1$. By (4.16) with $Du = f$, we get

$$\begin{aligned} &|D_y h(y, z) - D_y h(y', z)| \\ &= |Du(\psi^{-1}(y) + z) - Du(\psi^{-1}(y)) - Du(\psi^{-1}(y') + z) + Du(\psi^{-1}(y'))| \|D\psi^{-1}\|_0 \\ &\leq C_1 [Du]_\gamma |y - y'|^\delta |z|^{\gamma'}, \end{aligned}$$

for any $y, y' \in \mathbb{R}^d$. Let now $|z| > 1$; we find, for $y, y' \in \mathbb{R}^d$ with $|y - y'| \leq 1$,

$$|D_y h(y, z) - D_y h(y', z)| \leq C_2 [Du]_\gamma |y - y'|^\gamma \leq C_2 [Du]_\gamma |y - y'|^{\gamma-\gamma'}.$$

On the other hand, if $|y - y'| > 1$, $|z| > 1$,

$$|D_y h(y, z) - D_y h(y', z)| \leq 4 \|Du\|_0 |y - y'|^{\gamma-\gamma'}.$$

In conclusion, the second part of (2) is verified with $\delta = \gamma - \gamma'$ and

$$K_2(z) = C_3 \|Du\|_\gamma (1_{\{|z| \leq 1\}} |z|^{\gamma'} + 1_{\{|z| > 1\}}).$$

(note that $\int_{\mathbb{R}^d} K_2(z)^p \nu(dz) < \infty$, for any $p \geq 2$, since $2\gamma' > \alpha$). Since $C_b^\gamma(\mathbb{R}^d, \mathbb{R}^d) \subset C_b^{\gamma-\gamma'}(\mathbb{R}^d, \mathbb{R}^d)$, we deduce that both (1) and (2) hold with $\delta = \gamma - \gamma'$.

Applying [14, Theorem 3.4], we get that $y \mapsto Z_t^y$ is C^1 , P -a.s., and this proves our assertion. We finally note that [14, Theorem 3.4] also provides a formula for $H_t^y = DZ_t^y = DY_t^y$, i.e.,

$$\begin{aligned} H_t^y &= I + \lambda \int_0^t Du(\psi^{-1}(Z_s^y)) D\psi^{-1}(Z_s^y) H_s^y ds \\ &\quad + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} \left(D_y h(Z_{s-}^y, z) H_{s-}^y \right) \tilde{N}(ds, dz), \quad t \geq 0, y \in \mathbb{R}^d. \end{aligned}$$

The stochastic integral is meaningful, thanks to (2) in (4.15) and also to the fact that [14, assertion (3.10)] implies that, for any $t > 0$, $p \geq 2$, $\sup_{0 \leq s \leq t} E[|H_s|^p] < \infty$. The proof is complete. \blacksquare

Proof of Theorem 1.1. We may assume that $1 - \alpha/2 < \beta < 2 - \alpha$. We will deduce the assertion from Theorem 4.3.

Since $\alpha \geq 1$, we can apply Theorem 3.4 and find a solution $u_\lambda \in C_b^{1+\gamma}(\mathbb{R}^d, \mathbb{R}^d)$ to the resolvent equation (4.3) with $\gamma = \alpha - 1 + \beta \in (0, 1)$. By the last assertion of Theorem 3.4, we may choose λ sufficiently large in order that $\|Du\|_0 = \|Du_\lambda\|_0 < 1/3$. The crucial assumption about γ and α in Theorem 4.3 is satisfied. Indeed $2\gamma = 2\alpha - 2 + 2\beta > \alpha$ since $\beta > 1 - \alpha/2$. By Theorem 4.3 we obtain the result. \blacksquare

Remark 4.4. Thanks to Theorem 1.1 we may define a stochastic flow associated to (1.1). To this purpose, note that by (ii) we have $X_t^x = \xi_t(x)$, $t \geq 0$, $x \in \mathbb{R}^d$, P -a.s., where ξ_t is a homeomorphism from \mathbb{R}^d onto \mathbb{R}^d . Let ξ_t^{-1} be the inverse map. As in [14, Section 3.4], we set $\xi_{s,t}(x) = \xi_t \circ \xi_s^{-1}(x)$, $0 \leq s \leq t$, $x \in \mathbb{R}^d$.

The family $(\xi_{s,t})$ is a stochastic flow since verifies the following properties (P -a.s): (i) for any $x \in \mathbb{R}^d$, $(\xi_{s,t}(x))$ is a càdlàg process with respect to t and a càdlàg process with respect s ; (ii) $\xi_{s,t} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is an onto homeomorphism, $s \leq t$; (iii) $\xi_{s,t}(x)$ is the unique solution to (1.1) starting from x at time s ; (iv) we have $\xi_{s,t}(x) = \xi_{u,t}(\xi_{s,u}(x))$, for all $0 \leq s \leq u \leq t$, $x \in \mathbb{R}^d$, and $\xi_{s,s}(x) = x$.

Acknowledgements. The author is grateful to F. Flandoli for drawing his attention to a question by L. Mytnik which was the starting point of this work. He also thanks the Newton Institute (Cambridge) for good working conditions.

References

- [1] D. Applebaum : Lévy processes and stochastic calculus, Cambridge Studies in Advanced Mathematics 93, Cambridge University Press, 2004.
- [2] R.F. Bass : *Regularity results for stable-like operators*, J. Funct. Anal. **257** (2009), 2693-2722.
- [3] R.F. Bass and Z.Q. Chen : *Systems of equations driven by stable processes*, Probab. Theory Related Fields **134** (2006), 175-214.
- [4] G. Da Prato and A. Lunardi *On the Ornstein-Uhlenbeck operator in spaces of continuous functions*, J. Funct. Anal. **131** (1995), 94-114.
- [5] G. Da Prato and F. Flandoli : *Pathwise uniqueness for a class of SDE in Hilbert spaces and applications*, J. Funct. Anal. **259** (2010), 243-267.
- [6] A.M. Davie : *Uniqueness of solutions of stochastic differential equations*, Int. Math. Res. Notices IMRN, no. **24** (2007) Art. ID rnm124, 26 pp..
- [7] F. Flandoli, M. Gubinelli and E. Priola : *Well-posedness of the transport equation by stochastic perturbation*, Invent. Math. **180** (2010), 1-53.
- [8] E. Fedrizzi and F. Flandoli : *Pathwise uniqueness and continuous dependence for SDEs with nonregular drift*, Preprint Arxiv (2010) arXiv:1004.3485v1.
- [9] I. Gyöngy and T. Martinez : *On stochastic differential equations with locally unbounded drift*, Czechoslovak Math. J. (4) **51 (126)** (2001) 763-783.
- [10] N. Jacob : Pseudo differential operators and Markov processes. Vol. I. Fourier analysis and semigroups, Imperial College Press, London, 2001.
- [11] V. Kolokoltsov : *Symmetric stable laws and stable-like jump-diffusions*, Proc. London Math. Soc. (3) **80** (2000), 725-768.
- [12] N.V. Krylov : Lectures on elliptic and parabolic equations in Hölder spaces, Graduate Studies in Mathematics, 12, American Mathematical Society, Providence, RI, 1996.
- [13] N.V. Krylov and M. Röckner : *Strong solutions to stochastic equations with singular time dependent drift*, Probab. Theory Relat. Fields **131** (2005), 154-196.
- [14] H. Kunita : *Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms*, Real and stochastic analysis, Trends Math., Birkhäuser Boston, MA (2004), 305-373.
- [15] V.P. Kurenok : *Stochastic equations with time-dependent drift driven by Lévy processes*, J. Theoret. Probab. **20** (2007), 859-869.
- [16] A. Lunardi : Interpolation theory. Second edition, Lecture Notes, Scuola Normale Superiore di Pisa (New Series), Edizioni della Normale, Pisa, 2009.
- [17] J. Picard : *On the existence of smooth densities for jump processes*, Probab. Theory Related Fields **105** (1996), 481-511.
- [18] E. Priola : *Schauder estimates for a class of degenerate Kolmogorov equations*, Studia Math. **194** (2009), 117-153.
- [19] P. E. Protter : Stochastic Integration and Differential Equations. Second Edition, Stochastic Modelling and Applied Probability, Springer-Verlag, Berlin, 2004.

- [20] G. Samorodnitsky and M.S. Taqqu : Stable non-Gaussian random processes. Stochastic models with infinite variance, Stochastic Modeling, Chapman & Hall, New York, 1994.
- [21] K.I. Sato : Lévy processes and infinite divisible distributions, Cambridge University Press, Cambridge, 1999.
- [22] P. Sztonyk : *Regularity of harmonic functions for anisotropic fractional Laplacian*, Math. Nachr. **283** (2010), 289-311.
- [23] H. Tanaka, M. Tsuchiya and S. Watanabe : *Perturbation of drift-type for Lévy processes*, J. Math. Kyoto Univ. **14** (1974), 73-92.
- [24] A.J. Veretennikov : *Strong solutions and explicit formulas for solutions of stochastic integral equations*, Mat. Sb., (N.S.) **111 (153)** (3) (1980), 434-452.
- [25] X. Zhang : *Strong solutions of SDES with singular drift and Sobolev diffusion coefficients*, Stochastic Process. Appl. **115** (2005), 1805-1818.
- [26] X. Zhang : *Homeomorphism flows for non-Lipschitz stochastic differential equations with jumps*, Stochastic Process. Appl. **118** (2008), 2254-2268.
- [27] A.K. Zvonkin : *A transformation of the phase space of a diffusion process that removes the drift*, Mat. Sb. (N.S.) **93 (135)** (1974), 129-149.
- [28] J. Zabczyk : Topics in stochastic processes, Publication of Scuola Normale Superiore, Scuola Normale Superiore, Pisa, 2004.

Dipartimento di Matematica, Università di Torino
 via Carlo Alberto 10 10123
 Torino, Italy
 e-mail: enrico.priola@unito.it